A curve in the xy-plane will be said to be traversed in the positive sense when it is gone over in the direction indicated by the arrow in the figure. For the xz-plane the positive direction is taken as opposite to that indicated by the arrow. For the yz-plane it will be found necessary in the discussion of equation (2) to take the positive direction as indicated by the arrow in the figure. Areas are to be considered positive when their boundaries are traversed in the positive sense and negative when their boundaries are traversed in the negative sense. These definitions of the positive sense for the three coordinate planes are independent of the octants and seem to be the simplest for our present purpose.

By an inspection of the figure we see that the curves C and C_1 are traced by corresponding points in the same sense when $\partial P/\partial y < 0$ and in opposite senses when $\partial P/\partial y > 0$. If then we take into account the sign of the projective factor $\partial P/\partial y$ and the sense in which the curve C_1 is traversed when the curve C is gone over in the positive sense, we find that always

$$-\int_{(C)} Pdx = \int_A\int \frac{\partial P}{\partial y}\,dxdy.$$

ON A CERTAIN CLASS OF DETERMINANTS.

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Translated by permission from the *Rendiconti della R. Accademia delle Scienze Fisiche e Matematiche di Napoli* (3), volume XX, 1914.

The AMERICAN MATHEMATICAL MONTHLY for June, 1914, volume 21, page 184, contains the following question under the heading "A simple algebraic paradox."

Given two linear homogeneous complex equations

$$\begin{aligned}(a+bi)(p+qi)+(c+di)(r+si)&=0,\\(a'+b'i)(p'+q'i)+(c'+d'i)(r'+s'i)&=0.\end{aligned}\tag{1}$$

In order that these should be compatible it is necessary and sufficient that

$$\begin{vmatrix}(a+bi) & (c+di)\\(a'+b'i) & (c'+d'i)\end{vmatrix}=0.$$

This complex equation is equivalent to the *two* real equations

$$\begin{aligned}ac'+a'c&=bd'-b'd,\\ad'+bc'&=a'd+b'c.\end{aligned}\tag{1}$$

Both of these must be fulfilled if (1) is to subsist. On the other hand equations (1) are equivalent to

$$\begin{aligned}ap-bq+cr-ds&=0,\\bp+aq+dr+cs&=0,\\a'p-b'q+c'r-d's&=0,\\b'p+a'q+d'r+c's&=0.\end{aligned}$$

These are compatible when, and only when, a *single* equation is satisfied, namely

$$(2) \qquad \begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

Which answer is right?

How can a single condition be equivalent to two? This is the question asked by the American mathematician.

The answer is simple. The determinant (2) may be expressed as a sum of two squares, more specifically as the sum of the squares of the two right members of (1). Consequently the vanishing of (2) implies both of the equations (1).

But this remark suggests immediately a whole class of determinants (with real elements) *which have the remarkable property of being expressible as a sum of two squares.* The law of formation of these determinants is similar to that of the so-called skew symmetric determinants, with this difference however that the property of skew symmetry does not hold with respect to the elements of the determinant themselves, but with respect to certain partial matrices included in the determinant. The following determinant is representative of this type.

$$D = \begin{vmatrix} a_1' & \cdots & a_n' & b_1' & \cdots & b_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1^{(n)} & \cdots & a_n^{(n)} & b_1^{(n)} & \cdots & b_n^{(n)} \\ -b_1' & \cdots & -b_n' & a_1' & \cdots & a_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -b_1^{(n)} & \cdots & -b_n^{(n)} & a_1^{(n)} & \cdots & a_n^{(n)} \end{vmatrix}.$$

Such a determinant would be obtained in connection with a system of n linear homogeneous equations of the kind considered at the beginning of this note, by carrying out the discussion there given in analogous fashion.

Since the condition $D = 0$ must correspond to two relations, one might conclude *indirectly* that D *must be a sum of two squares.* But this fact may also be proved directly by the following method, in expounding which we shall confine ourselves to the case $n = 3$. The same method however is applicable for any value of n.

If we expand the determinant D (for $n = 3$) as a sum of products of the minors contained in the first n columns by their algebraic complements, and make use of the customary notation of the symbolic theory of n-ary forms, we find

$$\begin{aligned} D = {} & (a'a''a''')^2 + (a'a''b''')^2 + (a'a'''b'')^2 + (a''a'''b')^2 \\ & + (b'b''b''')^2 + (b'b''a''')^2 + (b'b'''a'')^2 + (b''b'''a')^2 \\ & + 2(a'a''b')(b'''a''a''') - 2(a'a''b'')(b'''a'a''') \\ & - 2(a'a'''b')(b''a''a''') + 2(b'b''a')(a'''b''b''') \\ & - 2(b'b''a'')(a'''b'b''') - 2(b'b'''a')(a''b''b'''). \end{aligned}$$

If now we make use of the identity

$$(a'a''b')(a''a'''b''') = (a'a''a''')(a''b'b''') + (a'a''b''')(b'a''a'''),$$

and of the analogous ones

$$\begin{aligned}
(a'a''b'')(b'''a'a''') &= (a'a''a''')(a'b''b''') - (a'a''b''')(a'b''a'''),\\
(a'a'''b')(b''a''a''') &= -\ (a'a''a''')(b'b''a''') - (a'b''a''')(b'a''a'''),\\
(b'b''a')(a'''b''b''') &= -\ (b'b''b''')(a'b''a''') + (b'b''a''')(a'b''b'''),\\
(b'b''a'')(a'''b'b''') &= (b'b''b''')(b'a''a''') - (b'b''a''')(b'a''b'''),\\
(b'b'''a')(a''b''b''') &= (b'b''b''')(a'a''b''') - (b'a''b''')(a'b''b'''),
\end{aligned}$$

the above expansion reduces to the sum of squares

$$[(a'a''a''') - (a'b''b''') + (b'a''b''') - (b'b''a''')]^2 \\ + [(b'b''b''') - (b'a''a''') + (a'b''a'') - (a'a''b''')]^2.$$

If we represent by (A) the matrix of the a's, by $(-A)$ that of the $-a$'s, and similarly by (B) and $(-B)$ the matrices of the b's and $-b$'s respectively, the matrix of the determinant D may be represented symbolically as follows:

$$\begin{Vmatrix} (A) & (B) \\ (-B) & (A) \end{Vmatrix}.$$

It still remains to study determinants whose matrices are of the type

$$\begin{Vmatrix} (A) & (B) & (C)\cdots \\ (-B) & (A) & (D)\cdots \\ (-C) & (-D) & (A)\cdots \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \end{Vmatrix}.$$

August, 1914.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY.

Analytic Geometry of Space. By VIRGIL SNYDER and C. H. SISAM. Henry Holt and Co., New York, 1914. xi+289 pages.

It is probable that in no branch of elementary mathematics has there been such need of a good, teachable book as in the analytic geometry of space. Books on this subject, designed for two half-year courses, are strangely lacking. Charles Smith's book on solid geometry gives fine results when used with a small class of picked students because, like so many other text-books from England, it forces the student who would get anything from it to think, to remember and to coördinate many branches of elementary mathematics. In other words, the book is